M3401 2009 Model Solutions

1. (a) $y'' - 2y' - 8y = e^{4x} \cosh x$

A.E. $m^2 - 2m - 8 = 0$

$\Rightarrow (m+2)(m-4) = 0$

$\Rightarrow m = -2$ or 4

$\therefore$ C.F. $y = Ae^{-2x} + Be^{4x}$

For P.I. try $y = A(x)e^{-2x} + B(x)e^{4x}$

$\Rightarrow y' = A'e^{-2x} - 2Ae^{-2x} + B'e^{4x} + 4Be^{4x}$

Pick $\underline{A'e^{-2x} + B'e^{4x} = 0}$ (A)

$\Rightarrow y' = -2Ae^{-2x} + 4Be^{4x}$

$\Rightarrow y'' = 4Ae^{-2x} - 2A'e^{-2x} + 16Be^{4x} + 4B'e^{4x}$

Sub in o.d.e.

$4Ae^{-2x} - 2A'e^{-2x} + 16Be^{4x} + 4B'e^{4x} + 4Ae^{-2x}$
$-8Be^{4x} - 8Ae^{-2x} - 8Be^{4x} = e^{4x}\cosh x$

$\Rightarrow \underline{-2A'e^{-2x} + 4B'e^{4x} = e^{4x}\cosh x}$ (B)

2(A) + (B) $\Rightarrow$ $6B'e^{4x} = e^{4x}\cosh x$

$\therefore B = \frac{1}{6}\sinh x$

4(A) − (B) $\Rightarrow$ $6A'e^{-2x} = -e^{4x}\cosh x$

$\therefore A = -\frac{1}{6}\int e^{6x}\cosh x\, dx$

i.e. $A = -\frac{1}{6}\int\left(\frac{e^{7x}}{2} + \frac{e^{5x}}{2}\right)dx$

$\therefore \quad A = -\frac{1}{6}\left(\frac{e^{7x}}{14} + \frac{e^{5x}}{10}\right)$

$\therefore$ P.I. is

$$y = -\frac{1}{6}\left(\frac{e^{5x}}{14} + \frac{e^{3x}}{10}\right) + \frac{1}{6}e^{4x}\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$\therefore \quad y = \frac{1}{14}e^{5x} - \frac{1}{10}e^{3x}$$

(b) Solve $\quad xy'' - (2x-1)y' - (8x+4)y = 0, \quad x > 0$

$$y_i = \int_{C_i} e^{xt} f(t)\,dt, \qquad i = 1,2$$

$$\Rightarrow \int_{C_i} e^{xt}\left[xt^2 - 2xt + t - 8x - 4\right] f(t)\,dt = 0$$

Exact differential of form $\frac{d}{dt}\left[e^{xt}g\right] = xge^{xt} + g'e^{xt}$

if

$$(t^2 - 2t - 8)f = g$$
$$(t-4)f = g'$$

$$\therefore \quad \frac{g'}{g} = \frac{(t-4)}{(t^2-2t-8)} = \frac{1}{(t+2)}$$

$$\therefore \quad \ln g = \ln (t+2)$$

$$\Rightarrow \quad g = (t+2)$$

$$\Rightarrow \quad f = \frac{1}{(t-4)}$$

$\therefore \quad y_i = \displaystyle\int_{C_i} e^{xt} \cdot \frac{1}{(t-4)} \, dt$ is a solution

provided that $\left[e^{xt} (t+2) \right]_{C_i} = 0$

$C_1 = t \in [-\infty, -2]$

$C_2 =$ closed curve around $t = 4$

$$\therefore \quad y_1 = \int_{-\infty}^{-2} \frac{e^{xt}}{(t-4)} \, dt$$

$$\& \quad y_2 = \oint_{\gamma} \frac{e^{xt}}{(t-4)} \, dt$$

$$\therefore \quad y_2 = \lim_{t \to 4} \left\{ \frac{(t-4) \, e^{xt}}{(t-4)} \right\} = e^{4x}$$

$\& \quad y_1(0) = \displaystyle\int_{-\infty}^{-2} \frac{dt}{(t-4)} \rightarrow -\infty$ is unbounded

2. (a) $$\frac{dy}{dx} = \frac{Cx + Dy}{Ax + By}$$

$(A+D)^2 \geqslant 4(AD-BC) > 0 \Rightarrow$ NODE

A node is stable if $(A+D) < 0$ and unstable if $(A+D) > 0$

$(A+D)^2 > 0 > 4(AD-BC) \Rightarrow$ SADDLE POINT

$4(AD-BC) > (A+D)^2 > 0 \Rightarrow$ SPIRAL POINT

A spiral point is stable if $(A+D) < 0$ and unstable if $(A+D) > 0$

$4(AD-BC) > 0$, $(A+D) = 0 \Rightarrow$ CENTRE

(b) A periodic trajectory is closed if and only if x and $y (= \dot{x})$ return to their values at t_0 after t has increased to $t_0 + T$.

$\therefore$ $x(t_0 + T) = x(t_0)$, t_0 arbitrary

& $\dot{x}(t_0 + T) = \dot{x}(t_0) = y$

$\Rightarrow$ $x(t)$ is periodic with period T

(c) Consider $\ddot{x} = x(x^3 - \lambda^3)$, $\lambda > 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x(x^3-\lambda^3)}{y}$$

Singular points are when $y = 0$ and $x(x^3 - \lambda^3) = 0$

i.e. $x = 0$ or λ

Consider singular point at $(0,0)$. Near to $(0,0)$

$$\frac{dy}{dx} \approx -\frac{\lambda^3 x}{y}$$

$$\therefore \quad \frac{y^2}{2} + \frac{\lambda^3 x^2}{2} = c \quad \Rightarrow \quad \text{CENTRE}$$

or/ $C = -\lambda^3, \ D = 0, \ A = 0, \ B = 1$

$$\Rightarrow \quad 4(AD - BC) = 4\lambda^3$$

$$(A + D) = 0$$

$$\Rightarrow \quad \text{CENTRE}$$

Consider singular point at $(\lambda, 0)$. Move s.p. to origin: $x = \lambda + X, \ y = Y$; $|X|, |Y| \ll 1$

$$\therefore \quad \frac{dY}{dX} = \frac{(\lambda + X)((\lambda + X)^3 - \lambda^3)}{Y}$$

$$= \frac{(\lambda + X)(\lambda^3 + 3\lambda^2 X + 3\lambda X^2 + X^3 - \lambda^3)}{Y}$$

$$\approx \frac{3\lambda^3 X}{Y}$$

$$\therefore \quad \frac{Y^2}{2} - \frac{3\lambda^3 X^2}{2} = c \quad \Rightarrow \quad \text{saddle point}$$

or/ $C = 3\lambda^3, \ D = 0, \ A = 0, \ B = 1$

$$\Rightarrow \quad 4(AD - BC) = -12\lambda^3$$

$$(A + D) = 0$$

$$\Rightarrow \quad \text{saddle point}$$

The equation through saddle point has locally

$$m = \frac{3\lambda^3 X}{m X}$$

$\therefore \; m^2 = 3\lambda^3$ i.e. $m = \pm\sqrt{3}\,\lambda^{3/2}$

And $\dfrac{dy}{dx} = 0$ when $x = 0$ & $x = \lambda$

except at s.p.

$\dfrac{dy}{dx} = \infty$ when $y = 0$ except at s.p.

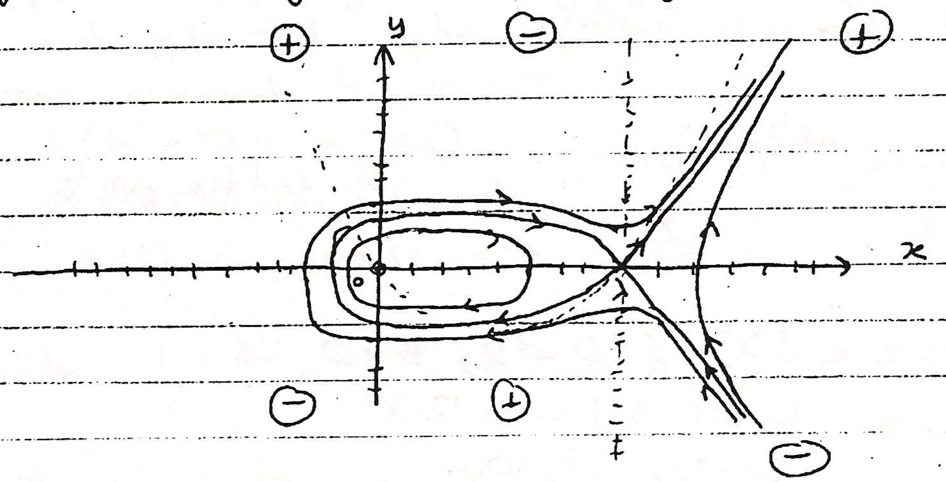
$\dfrac{dy}{dx} > 0$ when $x(x^3 - \lambda^3) > 0$, $y > 0$

or $x(x^3 - \lambda^3) < 0$, $y < 0$

$\dfrac{dy}{dx} < 0$ when $x(x^3 - \lambda^3) < 0$, $y > 0$

or $x(x^3 - \lambda^3) > 0$, $y < 0$

Trajectories are symmetric about $y = 0$



Solve phase equation:

$$\frac{y^2}{2} = \frac{x^5}{5} - \frac{\lambda^3 x^2}{2} + c$$

On separatrix

$$0 = \frac{\lambda^5}{5} - \frac{\lambda^5}{2} + c$$

$\therefore \quad c = \dfrac{3\lambda^5}{10}$ on separatrix

Also $y = u$ at $x = 0$

$$\Rightarrow \quad \frac{u^2}{2} = c$$

$$\therefore \quad \frac{y^2}{2} = \frac{x^5}{5} - \frac{\lambda^3 x^2}{2} + \frac{u^2}{2}$$

$\therefore$ Periodic motion occurs "under" the separatrix where $c < \dfrac{3\lambda^5}{10}$

$$\therefore \quad u^2 < \frac{3\lambda^5}{5}$$

Period is given by

$$T = \oint \frac{dx}{y} = \oint_\gamma \frac{dx}{\left(\frac{2x^5}{5} - \lambda^3 x^2 + u^2\right)^{1/2}}$$

where γ is one of the closed curves

3.

Consider $\ddot{x} + \epsilon \dot{x}(4x^2-1) + x = 0$

Write $\theta = nt$, $n = 2\pi/T$

So

$$n^2 \frac{d^2x}{d\theta^2} + \epsilon n \frac{dx}{d\theta}(4x^2-1) + x = 0$$

In one period of t, θ changes by 2π since $T = 2\pi/n$ $\therefore$ $x(\theta)$ is 2π-periodic

Subs in expansions into d.e.

$$(n_0 + \epsilon n_1 + \epsilon^2 n_2 + \dots)^2 (x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + \dots)$$
$$+ \epsilon(n_0 + \epsilon n_1 + \epsilon^2 n_2 + \dots)(x_0' + \epsilon x_1' + \epsilon^2 x_2' + \dots)(4[x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots]^2 - 1)$$
$$+ (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) = 0$$

<u>$O(1)$</u> $\quad n_0^2 x_0'' + x_0 = 0$

$$\therefore \quad x_0 = A_0 \cos\left(\frac{\theta}{n_0}\right) + B_0 \sin\left(\frac{\theta}{n_0}\right)$$

$x(0) = 0 \Rightarrow x_0(0) = 0 \quad \therefore \quad A_0 = 0$

$x(\theta) = x(\theta + 2\pi) \Rightarrow n_0 = 1$

$$\therefore \quad \underline{x_0 = B_0 \sin\theta}$$

<u>$O(\epsilon)$</u> $\quad 2n_0 n_1 x_0'' + n_0^2 x_1'' + n_0 x_0'(4x_0^2 - 1) + x_1 = 0$

$$\therefore \quad x_1'' + x_1 = -2n_1 x_0'' - x_0'(4x_0^2 - 1)$$

Now $x_0 = B_0 \sin\theta \Rightarrow x_0' = B_0 \cos\theta \Rightarrow x_0'' = -B_0 \sin\theta$

& $x_0'(4x_0^2 - 1) = B_0 \cos\theta (4B_0^2 \sin^2\theta - 1)$

Continuing from last line:

$$\text{Now} = 4B_0^3 \cos\theta \sin^2\theta - B_0 \cos\theta$$
$$= 4B_0^3 \cos\theta(1 - \cos^2\theta) - B_0 \cos\theta$$
$$= 4B_0^3 \cos\theta - 4B_0^3 \cos^3\theta - B_0 \cos\theta$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

i.e. $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

$$\therefore \quad x_0'(4x_0^2 - 1) = 4B_0^3\cos\theta - B_0^3\cos 3\theta - 3B_0^3\cos\theta - B_0\cos\theta$$

i.e. $x_0'(4x_0^2 - 1) = B_0^3\cos\theta - B_0\cos\theta - B_0^3\cos 3\theta$

$$\therefore \quad x_1'' + x_1 = 2n_1 B_0 \sin\theta - (B_0^3 - B_0)\cos\theta + B_0^3\cos 3\theta$$

Since $x_1 = A_1\cos\theta + B_1\sin\theta$ is the C.F. we must avoid $\sin\theta$, $\cos\theta$ terms on the r.h.s. to eliminate the non-periodic terms in the solution

$\therefore \quad 2n_1 B_0 = 0$ & $B_0^3 - B_0 = 0$

$\Rightarrow \quad n_1 = 0$ to avoid trivial solution

$\therefore \quad B_0(B_0^2 - 1) = 0 \Rightarrow B_0 = \pm 1$

Choose $B_0 = 1$, other choice gives solution of opposite phase

$$\therefore \quad x_1'' + x_1 = \cos 3\theta$$

Try P.I. of $x_1 = \lambda\cos 3\theta + \mu\sin 3\theta$

$\Rightarrow \quad x_1' = -3\lambda\sin 3\theta + 3\mu\cos 3\theta$

$\Rightarrow \quad x_1'' = -9\lambda\cos 3\theta + 9\mu\sin 3\theta$

$$\therefore \quad -9\lambda\cos 3\theta + 9\mu\sin 3\theta + \lambda\cos 3\theta + \mu\sin 3\theta = \cos 3\theta$$

$\therefore \quad \mu = 0$ & $\lambda = -\frac{1}{8}$

$$\therefore \quad x_1 = A_1 \cos\theta + B_1 \sin\theta - \frac{1}{8} \cos 3\theta$$

$$x_1(0) = 0 \quad \Rightarrow \quad 0 = A_1 - 1/8$$

$$\therefore \quad A_1 = \frac{1}{8}$$

$$\therefore \quad \underline{x_1 = \frac{1}{8}(\cos\theta - \cos 3\theta) + B_1 \sin\theta}$$

To find the equation at $O(\varepsilon^2)$ we now have

$$\begin{aligned}(1+\varepsilon^2 \Lambda_2)^2 (x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \dots) \\ + \varepsilon (1+\varepsilon^2\Lambda_2)(x_0' + \varepsilon x_1' + \varepsilon^2 x_2' + \dots) \\ * (4[x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2 x_1 + 2\varepsilon^2 x_0 x_2] - 1) \\ + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) = 0\end{aligned}$$

$$\therefore \quad x_2'' + 2\Lambda_2 x_0'' + x_1'[4x_0^2 - 1] + x_0'[8x_0 x_1] + x_2 = 0$$

$$\text{i.e.} \quad \underbrace{x_2'' + x_2}_{1} = \underbrace{-2\Lambda_2 x_0''}_{1} - \underbrace{x_1'[4x_0^2 - 1]}_{1} \underbrace{- 8x_0 x_0' x_1}_{1}$$

4. Define $F(x) = \int_0^x f(t)\,dt$

(a)

& put

$$y = \dot{x} + \varepsilon F(x)$$

i.e. $\dot{x} = y - \varepsilon F(x)$

Sub in the d.e. to get, after noting

$$\ddot{x} = \dot{y} - \varepsilon \frac{d}{dt} F = \dot{y} - \varepsilon \frac{dF}{dx}\dot{x} = \dot{y} - \varepsilon f \dot{x}, ($$

to find

$$\dot{y} - \varepsilon f \dot{x} + \varepsilon f \dot{x} + x = 0$$

$$\therefore \quad \dot{y} = -x$$

Now $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{x}{\varepsilon F - y}$

(b) If a periodic solution exists then there is a unique closed curve in (x, y) plane such that

$$\oint_\gamma F(x)\,dy = 0$$

$$\Rightarrow \oint F(x) \frac{dy}{dx}\,dx = 0$$

$$\Rightarrow \underbrace{\left[F(x)\, y\right]_\gamma}_{=0} - \oint_\gamma y\, F'(x)\,dx = 0$$

$$\Rightarrow \oint_\gamma y\, f(x)\,dx = 0$$

Q.E.D.

(c) Now $f = 2x^2 - 8$

$$\therefore \quad F = \int_0^x (2t^2 - 8)\,dt = \frac{2x^3}{3} - 8x$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{\epsilon\left(\frac{2x^3}{3} - 8x\right) - y}$$

Write $y = \epsilon z$, so

$$\frac{dz}{dx} = \frac{x}{\epsilon^2\left[\left(\frac{2x^3}{3} - 8x\right) - z\right]}$$

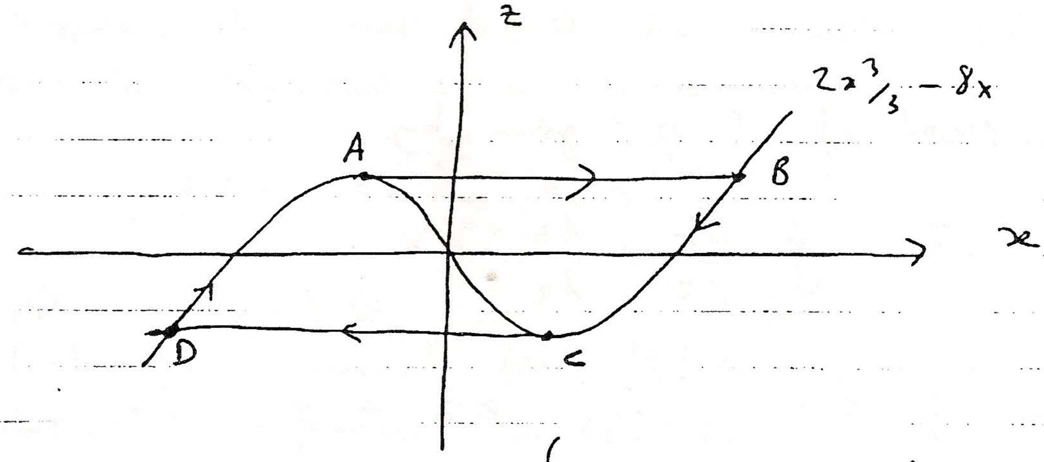
$$\therefore \quad \left[\left(\frac{2x^3}{3} - 8x\right) - z\right]\frac{dz}{dx} = x/\epsilon^2$$

If $\epsilon \gg 1$ then r.h.s is zero to leading order. We seek a periodic solution, so x & z remain finite, since the trajectory is closed.

$$\therefore \quad \text{Either} \quad \left(\frac{2x^3}{3} - 8x\right) - z = 0 \quad \text{or} \quad \frac{dz}{dx} = 0$$

i.e. $z = \dfrac{2x^3}{8} - 8x$ or $\dfrac{dz}{dx} = 0$

$\therefore$ For $\varepsilon >> 1$ the plane plane looks like



$$z = \frac{2x^3}{3} - 8x$$

$$\therefore \frac{dz}{dx} = 2x^2 - 8 = 0 \Rightarrow x = \pm 2$$

If $x = 2$ then $z = \frac{16}{3} - 16 = \frac{-32}{3}$

If $x = -2$ then $z = -\frac{16}{3} + 16 = \frac{32}{3}$

$\frac{2x^3}{3} - 8x = \frac{32}{3}$ when $x = -2$ or 4 from hint

$\frac{2x^3}{3} - 8x = \frac{-32}{3}$ when $x = 2$ or -4 from hint

$\therefore$ A has co-ords $(-2, 32/3)$

B has co-ords $(4, 32/3)$

C has co-ords $(2, -32/3)$

D has co-ords $(-4, -32/3)$

The period of T is given by

$$T = \oint_\gamma \frac{dt}{dz} \cdot \frac{dz}{dx} \cdot dx$$

Now $\dot{y} = -x \quad \therefore \quad \varepsilon \dot{z} = -x$

&

$$T = \left\{ \int_A^B + \int_B^C + \int_C^D + \int_D^A \right\} \left\{ -\frac{\varepsilon}{x} \cdot \frac{dz}{dx} \, dx \right\}$$

i.e. $$T = \int_4^2 -\frac{\varepsilon}{x} (2x^2 - 8) \, dx - \int_{-4}^{-2} \frac{\varepsilon}{x} (2x^2 - 8) \, dx$$

Since $$\int_A^B = \int_C^D = 0$$

ie. $$T = \left[-\varepsilon (x^2 - 8 \ln x) \right]_4^2 - \left[\varepsilon (x^2 - 8 \ln x) \right]_{-4}^{-2}$$

i.e. $$T = -\varepsilon (4 - 8 \ln 2) + \varepsilon (16 - 8 \ln 4) - \left[\varepsilon (4 - 8 \ln 2) - \varepsilon (16 - 8 \ln 4) \right]$$

ie. $$\underline{T = \varepsilon (24 - 16 \ln 2)}$$

& the amplitude $A = \max (|x(t)|) = 4$

5. Suppose that near $t=0$ the function $f(t)$

(a) can be expanded in a power series

$$f(t) = \sum_{n=0}^{\infty} a_n t^{\lambda_n}$$

with $a_0 \neq 0$
which is convergent for $|t| < t_0$,
where the sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$ is
increasing with n and $\lambda_0 > -1$.

Suppose also when $t \geq t_0$

$$|f(t)| < B e^{ct}$$

with $c > 0$

Then as $x \to +\infty$

$$\int_0^T e^{-xt} f(t)\,dt \sim \sum_{n=0}^{\infty} \frac{a_n (\lambda_n)!}{x^{\lambda_n + 1}}$$

(b)

$$I = \int_a^b e^{x\phi(t)} f(t)\,dt = \left\{ \int_a^{t_0} + \int_{t_0}^b \right\} e^{x\phi(t)} f(t)\,dt = I_1 + I_2$$

For $\left\{ \begin{array}{l} a \leq t \leq t_0 \\ t_0 \leq t \leq b \end{array} \right\}$ let $-u = \phi(t) - \phi(t_0)$

$$\therefore I_1 = -e^{-x\phi(t_0)} \int_0^{\alpha} e^{-xu} f(u) \frac{dt}{du}\,du$$

$$\& \ I_2 = e^{-x\phi(t_0)} \int_0^{\beta} e^{-xu} f(t(u)) \frac{dt}{du} \, du$$

Near $u = 0$, $t \sim t_0$

$$\therefore \ -u = \phi(t) - \phi(t_0)$$
$$\simeq \phi(t_0) + \underbrace{(t-t_0)\phi'(t_0)}_{=0} + \frac{(t-t_0)^2}{2}\phi''(t_0) - \phi(t_0)$$

$$\therefore \ -u \approx \frac{(t-t_0)^2}{2}\phi''(t_0)$$

$$\therefore \ t = t_0 \mp \left(\frac{-2u}{\phi''(t_0)}\right)^{1/2}$$

for $\begin{cases} a \le t \le t_0 \\ t_0 \le t \le b \end{cases}$

$$\therefore \ \frac{dt}{du} = \mp \frac{1}{2}\left(\frac{-2}{\phi''(t_0)}\right)^{1/2} u^{-1/2}$$

for $\begin{cases} a \le t \le t_0 \\ t_0 \le t \le b \end{cases}$

Cf. Watson's lemma: $a_0 = \mp e^{x\phi(t_0)}\left(\frac{-2u}{\phi''(t_0)}\right)^{1/2} f(t_0)$

$$\lambda_0 = -1/2$$

$\therefore$

$$I \sim 2 e^{x\phi(t_0)} f(t_0) \cdot \frac{1}{2} \cdot \left(\frac{-2}{\phi''(t_0)} \right)^{1/2} \frac{(-1/2)!}{x^{1/2}}$$

i.e. $$I \sim e^{x\phi(t_0)} f(t_0) \left(\frac{-2\pi}{\phi''(t_0) x} \right)^{1/2} \text{ as } x \to +\infty$$

If $t_0 = a$ or $t_0 = b$ result is divided by 2

(c) Method of stat. phase says

$$\int_a^b e^{ix\phi(t)} f(t)\,dt \sim \exp\left[ix\phi(t_0) \pm \frac{i\pi}{4} \right] f(t_0) \left(\frac{2\pi}{x|\phi''(t_0)|} \right)^{1/2}$$

as $x \to +\infty$ with $\pm$ sign taken according as $\phi''(t_0) \gtrless 0$ respectively

Here $\phi(t) = \cos t$

$\Rightarrow \phi'(t) = -\sin t$

$\Rightarrow \phi''(t) = -\cos t$

$\therefore \phi' = 0$ at $t = 0, \pi$

$\therefore \phi''(0) = -1 \Rightarrow$ max with $\phi(0) = 1$

& $\phi''(\pi) = +1 \Rightarrow$ min with $\phi(\pi) = -1$

$$\therefore I \sim \exp\left[ix - \frac{i\pi}{4} \right] \left(\frac{2\pi}{x} \right)^{1/2} + \exp\left[-ix + \frac{i\pi}{4} \right] \left(\frac{2\pi}{x} \right)^{1/2}$$

Take real part on both sides & use fact that cos is even

$$\int_{-\pi/2}^{3\pi/2} \cos(x\cos(t))\,dt \sim 2\cos\left(x-\frac{\pi}{4}\right)\left(\frac{2\pi}{x}\right)^{1/2}$$